J. DIFFERENTIAL GEOMETRY
9 (1974) 329-345

CARLEMAN-TYPE ESTIMATES AND THE NEUMANN PROBLEM

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Introduction

In the theory of overdetermined systems of partial differential equations the essential tool is the construction of the Spencer sequence. Namely, with a given operator $D: E \to F$ we can associate the sequence

$$0 \longrightarrow \Theta \xrightarrow{j_m} C^0 \xrightarrow{D^0} C^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} C^n \longrightarrow 0 \tag{0.1}$$

where Θ is the solution sheaf, D^l are first order differential operators, and C^l are sheaves of certain vector bundles [5]. This sequence reflects many of the properties of the differential operator considered. For example, an operator is elliptic if the symbol sequence

$$0 \longrightarrow C^0 \xrightarrow{\sigma_\xi(D)} C^1 \xrightarrow{\sigma_\xi(D)} C^2 \longrightarrow \cdots \xrightarrow{\sigma_\xi(D)} C^n \longrightarrow 0 \tag{0.2}$$

is exact for all real nonzero covectors ξ.

B. MacKichan studied in his Stanford thesis [4] a metric condition called the δ-estimate. This condition is shown to imply

$$-\sum_{i,j} {}^{\Omega}\langle A_j^*\partial_i u, A_i^*\partial_j u\rangle_\varphi \le c\, {}^{\Omega}\Big\|\sum_j B_j\partial_j u\Big\|_\varphi^2 , \tag{0.3}$$

the meaning of which will be explained in the sequel. Thus we see that the δ-estimate can be used conveniently to control a certain second order term for facilitating our computation.

In this paper we will assume that the operator $D: E \to F$ considered is elliptic and satisfies the δ-estimate. MacKichan proved that the Neumann problem is solvable in the Euclidean case with constant coefficients, and indicated without proof that it is solvable in the non-Euclidean case with variable coefficients.

Later Sweeney [6] showed that in the Euclidean case with variable coefficients the Neumann problem is solvable on a sufficiently small disc, and the harmonic spaces vanish in positive degrees.

Communicated by D. C. Spencer, January 24, 1973.

We shall extend this result of Sweeney in several directions. In Part I, we generalize to the Spencer sequence some of the Carleman-type estimates given by Hörmander [1] for the $\bar{\partial}$-operator. §§ 1 and 2 are concerned with the Euclidean case. In § 3 we examine the constant coefficient case and derive an inequality:

$$(0.4) \qquad \sum_{i,j}{}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_{\varphi} \leq c[{}^{\Omega}\|D^* u\|_{\varphi}^2 + {}^{\Omega}\|Eu\|_{\varphi}^2] ,$$

which holds true in a convex open set $\Omega \subset \boldsymbol{R}^n$. This main inequality (0.4) is applied to get existence and approximation theorems. In particular, if an element u is the coboundary of an element v, we show how v, properly chosen, can be bounded in terms of u. In § 4 we consider the non-Euclidean case. In §§ 2 and 4, the domain has to be shrunk in order to get rid of the error terms.

In Part II, we establish the estimates for the Neumann problem on a compact manifold with smooth boundary. First we summarize the D-Neumann problem in § 1, and point out that it is sufficient to have the Kohn-Nirenberg estimate. Then in § 2 we use the result of Part I to obtain this estimate in the whole domain by a partition of unity argument. We also examine the "local convexity" condition required on Ω, and try to express it by an invariant form. In § 3 we try to investigate a question raised in § 2 concerning the Levi form. However we are unable to determine an intrinsically defined Levi form in the general case, which, if positive definite, would ensure the validity of our estimates, and indeed no such form seems to exist. We give an example in which the required convexity on a bounded domain is stronger than that expressed by the Levi form. In place of the Levi form we use a form which is defined in terms of the given operator by means of a metric. If this form is positive definite, the desired estimate for the Neumann problem is shown to be valid. The problem of the existence of an intrinsically defined Levi form remains open (even though its positive-definiteness will not always be sufficient for the solvability of the Neumann problem).

PART I

1. Euclidean case with constant coefficients

Let Ω be an open set in $\boldsymbol{R}^n$ with compact closure and smooth boundary, that is, there is a smooth function $r(x)$ which defines the boundary $\partial\Omega$ of Ω, such that $r(x) < 0$ inside Ω, $r(x) = 0$ on $\partial\Omega$, $r(x) > 0$ outside Ω and such that $|dr| = 1$ on the boundary.

We are considering the Spencer sequence (0.1) which is elliptic and satisfies the δ-estimate. For convenience we shall write D^{l-1} as D, D^l as E, and write the formal adjoint $(D^{l-1})^*$ as D^*. First we assume that D and E are of the following forms:

$$(1.1)\qquad D = \sum_j A_j \frac{\partial}{\partial x_j} + A_0 = \sum_j A_j \partial_j + A_0 \,, \qquad E = \sum_j B_j \partial_j + B_0 \,,$$

where A_j, A_0 and B_j, B_0 are constant matrices. Moreover, we consider a real valued smooth function φ on Ω. Applying Green's formula to elements u in C^{l-1}, v in C^l, and defining

$$(1.2)\qquad {}^{\Omega}\langle Du, v\rangle_\varphi = \int_\Omega \langle Du, v\rangle_x e^{-\varphi} dV \,, \qquad \text{where } x \in \Omega \,,$$

we may conclude that:

$$(1.3)\qquad D^* = -\sum_j A_j^* \delta_j + A_0^* \qquad \text{with } \delta_j v = \partial v/\partial x_j - v(\partial\varphi/\partial x_j) \,,$$

and an element v belonging to the domain of D^* must satisfy

$$(1.4)\qquad \sum_j (\partial_j r) A_j^* v \equiv 0 \qquad \text{on } \partial\Omega \,.$$

Following a theorem in [4, pp. 115–116] and observing that in our case we have to consider

$$(1.5)\qquad (\delta_j \partial_i - \partial_i \delta_j) u = \frac{\partial^2 \varphi}{\partial x_i \partial x_j} u$$

when we interchange ∂_i and δ_j, we obtain the following inequality (1.6) which is a Carleman-type estimate.

Theorem 1. *Let Ω be an open set in $\boldsymbol{R}^n$ with compact closure and smooth boundary, and let D and E be defined by* (1.1). *Then we can find a constant c independent of u and φ such that*

$$(1.6)\qquad \sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u \right\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u, A_j^* u \right\rangle_\varphi \\ \leq c[{}^{\Omega}\|D^* u\|_\varphi^2 + {}^{\Omega}\|Eu\|_\varphi^2 + {}^{\Omega}\|u\|_\varphi^2] \,.$$

If Ω is a ball of radius R, we may take $r(x) = (\sum_i x_i^2)^{\frac{1}{2}} - R$, so that we have on $\partial\Omega$:

$$\partial_i \partial_j r = -\frac{1}{R}(\partial_i r)\partial_j r \quad \text{if} \quad i \neq j \,, \qquad (\partial_i)^2 r = \frac{1}{R}(1 - (\partial_i r)^2) \,.$$

Thus

$$\sum_{i,j} {}^{\partial\Omega}\langle(\partial_i\partial_j r)A_i^* u, A_j^* u\rangle_\varphi$$

$$(1.7) \qquad = \frac{1}{R}\sum_i {}^{\partial\Omega}\langle A_i^* u, A_i^* u\rangle_\varphi - \frac{1}{R}\sum_{i,j} {}^{\partial\Omega}\langle(\partial_i r)A_i^* u, (\partial_j r)A_j^* u\rangle_\varphi$$

$$= \frac{1}{R}\sum_i {}^{\partial\Omega}\|A_i^* u\|_\varphi^2 \geq c' \, {}^{\partial\Omega}\|u\|_\varphi^2 \geq 0 ,$$

which comes from the symbol surjectivity of the Spencer sequence [6, p. 357]. Substituting (1.7) into (1.6) we obtain

$$(1.8) \quad \sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi + c' \, {}^{\partial\Omega}\|u\|_\varphi^2 \leq c[{}^{\Omega}\|D^* u\|_\varphi^2 + {}^{\Omega}\|Eu\|_\varphi^2 + {}^{\Omega}\|u\|_\varphi^2] .$$

Taking further $\varphi = 0$, we get the Kohn-Nirenberg inequality

$$(1.9) \qquad {}^{\partial\Omega}\|u\|^2 \leq c[{}^{\Omega}\|D^* u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2] .$$

In the next section we shall show that the term ${}^{\Omega}\|u\|_\varphi^2$ appearing in (1.8) is in some sense not essential.

2. Euclidean case with variable coefficients

We shall consider now (1.1) with variable matrices A_j, A_0, B_j, B_0 depending on $x \in \Omega$ as coefficients. For convenience we shall consider in this section the unit ball, and denote it by Ω. Instead of D and E, we shall consider new operators D_ε and E_ε which are defined as follows:

$$(2.1) \quad D_\varepsilon = \sum_j A_j(\varepsilon x)\partial_j + \varepsilon A_0(\varepsilon x) , \qquad E_\varepsilon = \sum_j B_j(\varepsilon x)\partial_j + \varepsilon B_0(\varepsilon x) .$$

Then

$$(2.2) \qquad D_\varepsilon^* = -\sum_j A_j^*(\varepsilon x)\delta_j - \varepsilon\left(\sum_j \frac{\partial A_j^*}{\partial x_j}(\varepsilon x) - A_0^*(\varepsilon x)\right) .$$

Following a parallel treatment as in [6] (and the details of which will be carried out in § 4), we have

$$(2.3) \quad \begin{aligned} &\sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi \\ &\leq c[{}^{\Omega}\|D_\varepsilon^* u\|_\varphi^2 + {}^{\Omega}\|E_\varepsilon u\|_\varphi^2] + \varepsilon c \, {}^{\Omega}\|u\|_\varphi^2 + \varepsilon^{\Omega}\langle P_\varepsilon u, u\rangle_\varphi , \end{aligned}$$

where P_ε is a first order differential operator given by

$$P_\varepsilon = \left(-2\sum_j A_j A_0^* \partial_j - 2\sum_{i,j} \frac{\partial A_j}{\partial x_i} A_i^* \partial_j\right) \tag{2.4}$$
$$+ \varepsilon\left(-2\sum_j A_j \frac{\partial A_0^*}{\partial x_j} - A_0 A_0^* - \sum_{i,j} \frac{\partial A_i}{\partial x_j}\frac{\partial A_j^*}{\partial x_i}\right).$$

We can control P_ε by two lemmas in [6], and obtain

$$|{}^{\Omega}\langle P_\varepsilon u, u\rangle_\varphi| \le c[{}^{\Omega}\|D_\varepsilon^* u\|_\varphi^2 + {}^{\Omega}\|E_\varepsilon u\|_\varphi^2 + M(M+1)\, {}^{\Omega}\|u\|_\varphi^2 + {}^{\partial\Omega}\|u\|_\varphi^2] , \tag{2.5}$$

where we suppose that M is a fixed number such that

$$\sup_j |\partial\varphi/\partial x_j| \le M . \tag{2.6}$$

Choosing φ to satisfy (2.6) and to be strongly convex, namely,

$$\sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi \ge c\, {}^{\Omega}\|u\|_\varphi^2 , \tag{2.7}$$

we obtain the following theorem by shrinking the ball.

Theorem 2. *On a sufficiently small ball Ω, if we choose φ to satisfy* (2.6) *and* (2.7), *then we have the Carleman-type estimate*:

$$\sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi$$
$$\le c[{}^{\Omega}\|D^* u\|_\varphi^2 + {}^{\Omega}\|Eu\|_\varphi^2] ,$$

where c is a constant depending on M and of the form

$$c = constant/[1 - \varepsilon M(M+1)] . \tag{2.8}$$

Recalling the remark following (1.9) in § 1, if we are only interested in a small Ω, then the term ${}^{\Omega}\|u\|_\varphi^2$ can be absorbed. This is true for the Euclidean case. But we shall show in § 4 that this is no longer true in the non-Euclidean case. When Ω is an open subset with boundary in a manifold, there will be some term ${}^{\Omega}\|u\|_\varphi^2$ which cannot be absorbed, even when we consider a very small Ω. However, when the constant c is independent of u and φ, as is in the constant coefficient case, there is another method to take away the term ${}^{\Omega}\|u\|_\varphi^2$ in (1.9). This we shall show in the next section.

3. Applications of the Carleman-type estimate

In this section we shall transcribe the method of Hörmander [1] to obtain corresponding results for elliptic operators satisfying the δ-estimate.

We shall consider (1.6) on a convex domain Ω, that is, such that

(3.1)
$$\sum_{i,j}{}^{\partial\Omega}\left\langle\frac{\partial^2 r}{\partial x_i \partial x_j}A_i^* u, A_j^* u\right\rangle_\varphi \geq 0 .$$

Therefore this term can be dropped. Since c is a constant independent of u, φ, if we replace φ by $\tau\varphi$ for some large positive number τ, we get a similar inequality with the same c. Now we are considering a strongly convex φ. The inequality (2.7) is still true if we replace φ by $\tau\varphi$. Thus we obtain, by renaming the constants,

(3.2)
$$\tau {}^{\Omega}\|u\|_{\tau\varphi}^2 \leq c[{}^{\Omega}\|D^*u\|_{\tau\varphi}^2 + {}^{\Omega}\|Eu\|_{\tau\varphi}^2 + {}^{\Omega}\|u\|_{\tau\varphi}^2] .$$

If we choose $\tau > 2c$, we obtain formula (0.4) by replacing $\tau\varphi$ by φ. Next we shall transform $(\partial^2\varphi/\partial x_i \partial x_j)$ into its simplest form. Let $\lambda_u = e^k$ be the smallest eigenvalue, and consider the operator $T = e^{k/2}I$. Then we have

$$\sum_{i,j}{}^{\Omega}\left\langle\frac{\partial^2 \varphi}{\partial x_i \partial x_j}A_i^* u, A_j^* u\right\rangle_\varphi = \sum_i \lambda_i^\Omega \langle A_i^* u, A_i^* u\rangle_\varphi$$
$$\geq \sum_i {}^{\Omega}\langle A_i^* e^{k/2}u, A_i^* e^{k/2}u\rangle_\varphi \geq c' \, {}^{\Omega}\|Tu\|_\varphi^2 .$$

Combining (0.4) and (3.3) we get

(3.4)
$$\|Tu\|^2 \leq c[\|D^*u\|^2 + \|Eu\|^2] .$$

This operator T is closed, densely defined and linear in C^l. Moreover, $T^* = T$. Thus, if $g = T^*h$, then $h = T^{-1}g = e^{-k/2}g$, and we can apply a classical theorem in the Hilbert space theory [1] to obtain

Theorem 3. *Let Ω be an open set in $\boldsymbol{R}^n$ with a smooth boundary satisfying* (3.1), *φ be a suitable smooth strongly convex functon in Ω, and e^k be the lowest eigenvalue of the matrix $(\partial^2\varphi/\partial x_i \partial x_j)$. For every $u \in C^l(\Omega, \varphi)$ satisfying the boundary condition* (1.4) *and $Eu = D^l u = 0$, and*

$$\int_\Omega \langle u, u\rangle e^{-(\varphi+k)}dV < \infty ,$$

then we can find a $v \in C^{l-1}(\Omega, \varphi)$ such that $Dv = D^{l-1}v = u$ and

(3.5)
$$\int_\Omega \langle v, v\rangle e^{-\varphi}dV \leq c\int_\Omega \langle u, u\rangle e^{-(\varphi+k)}dV ,$$

where $C^l(\Omega, \varphi)$ means the elements in $C^l(\Omega)$ whose φ-norm is finite.

Later we shall use the notation $C^l(\Omega, \mathrm{loc})$ to represent those elements in C^l which are integrable on every compact subset of Ω. Of course, if $u \in C^l(\Omega, \varphi)$, then $u \in C^l(\Omega, \mathrm{loc})$.

The assumption concerning the boundary smoothness in the above theorem can be removed by using the strong inequality (3.5), and we can express Theorem 3 in a more applicable version:

Theorem 3′. *If Ω is a convex open set in $\boldsymbol{R}^n$, and $u \in C^l(\Omega, \text{loc})$ satisfies the boundary condition and the integrability condition $Eu = 0$, then there exists $v \in C^{l-1}(\Omega, \text{loc})$ such that $Dv = u$ and*

$$\int_\Omega \langle v, v\rangle e^{-\varphi} dV \leq c \int_\Omega \langle u, u\rangle e^{-\varphi} dV .$$

We shall make an application of Theorem 3′ to obtain a theorem concerning the vanishing of the cohomology, and moreover to establish an estimate between the cocycle and the cochain of which it is the coboundary. Given a convex Ω in $\boldsymbol{R}^n$ with compact closure and smooth boundary, we cover Ω with a finite system of convex covering $\{\Omega_j\}$, namely, each Ω_j is convex. Consider a suitable partition of unity $\{\chi_j\}$ subject to this covering. If s is an integer ≥ 0, we denote by $W^s(Z_l(\{\Omega_j\}, \varphi))$ the set of all alternating cochains $c = \{c_\alpha\}$ where $\alpha = (\alpha_0, \cdots, \alpha_s)$ is an $(s+1)$-tuple of positive integers, $c_\alpha \in C^l(\Omega_\alpha), \Omega_\alpha = \Omega_{\alpha_0} \wedge \cdots \wedge \Omega_{\alpha_s}, D^l c_\alpha = 0$ and

$$\|c\|_\varphi^2 = \sum_\alpha \int_{\Omega_\alpha} \langle c_\alpha, c_\alpha\rangle e^{-\varphi} dV < \infty . \tag{3.6}$$

As usual we define the coboundary operator δ from W^s to W^{s+1} by

$$(\delta c)_{\alpha_0 \cdots \alpha_{s+1}} = \sum_{k=0}^{s+1} (-1)^k c_{\alpha_0 \cdots \hat{\alpha}_k \cdots \alpha_{s+1}} . \tag{3.9}$$

Theorem 4. *Assume that the partition of unity $\{\chi_j\}$ subordinate to $\{\Omega_j\}$ is such that $|\sum_j [D^l, \chi_j]| \leq$ constant. Let φ be a strongly convex function. For every $c \in W^s(Z_l(\{\Omega_j\}, \varphi))$ with $\delta c = 0, s \geq 1$, one can then find a cochain $c' \in W^{s-1}(Z_l(\{\Omega_j\}, \varphi))$ such that $\delta c' = c$ and $\|c'\|_\varphi \leq K \|c\|_\varphi$, where the constant K does not depend on c.*

Sketch of proof. If we set $b_\alpha = \sum_j \chi_j c_{j,\alpha}$ for $|\alpha| = s$, it is easy to show that $\delta b = c$. Although $D^l b$ is not necessary equal to 0, we have $\delta D^l b = 0$. Consideration of the case $s = 1$ and then induction enable us to find a suitable error term of b, so that after correction we get $\delta b' = c$ and $b' \in W^{s-1}$ and the inequality indicated is satisfied.

A quite complicated argument such as that in [1] can yield the following important approximation theorem. We denote by K^C the complement of K. A compact subset K of Ω is said to be convex with respect to Ω if for every $x \in \Omega \cap K^C$ there is a convex function ψ in Ω such that $\psi(x) > 0$ but $\psi < 0$ in K. For such K we have

Theorem 5. *Let Ω be an open convex set in $\boldsymbol{R}^n$, and K be a compact subset of Ω which is convex with respect to Ω. Let $u \in C^{l-1}(K, 0)$, and let $Du = 0$ on K in the strong sense that*

$$^K\langle u, D_0^* w\rangle = \int_K \langle u, D_0^* w\rangle dV = 0$$

for every $w \in C^l(\Omega, 0)$ *such that* $w = 0$ *outside* K *and* $D_0^* w \in C^{l-1}(\Omega, 0)$. *Then one can approximate* u *arbitrarily closely in* $C^{l-1}(K, 0)$ *by elements* $u' \in C^{l-1}(\Omega, \text{loc})$ *such that* $Du' = 0$.

4. Carleman-type estimate for the non-Euclidean case

Let M be a real Riemannian manifold with metric (g_{ij}). Then the term C^l in the Spencer sequence is a fibre bundle on M, and we let (h_{ij}) be a metric along the bundle. Consider an open submanifold Ω in M with smooth boundary. At each point x in Ω, we have a coodinate neighborhood U with local coodinate $(x^1, \cdots, x^n)$. Let u, v be bundle-valued differential forms in $\Omega \cap U$, and φ a smooth function in Ω. If q is the fibre dimension of C^l, the form of u can be written as $u = (u^1, \cdots, u^q)$, where

$$u^i = \sum_{\alpha_1, \cdots, \alpha_l} u^i_{\alpha_1 \cdots \alpha_l}(x) dx^{\alpha_1} \cdots dx^{\alpha_l} ,$$

and we may define

$${}^{\Omega \cap U}\langle u, v \rangle_\varphi = \sum \int_{\Omega \cap U} h_{ij} g^{\alpha_1 \beta_1} \cdots g^{\alpha_l \beta_l} u^i_{\alpha_1 \cdots \alpha_l} v^j_{\beta_1 \cdots \beta_l} \sqrt{g}\, e^{-\varphi} dx .$$

The Green's formula is easily seen to be

$$(4.1) \qquad {}^{\Omega \cap U}\left\langle \frac{\partial u}{\partial x_k}, v \right\rangle_\varphi = - {}^{\Omega \cap U}\langle u, \delta_k v \rangle_\varphi + {}^{\partial \Omega \cap U}\left\langle \frac{\partial r}{\partial x_k} u, v \right\rangle_\varphi + {}^{\Omega \cap U}\langle u, \sigma_k v \rangle_\varphi ,$$

where δ_k is defined in (1.3), and

$$(4.2) \qquad \sigma_k = \left(\frac{\partial H^{-1}}{\partial x_k} H + \frac{\partial G^{-1}}{\partial x_k} G + \cdots + \frac{\partial G^{-1}}{\partial x_k} G - \frac{1}{2} \frac{\partial \log g}{\partial x_k} \right)$$

with $H = (h_{ij})$ and $G = (g_{ij})$. We see that σ_k thus defined is independent of u, v, and depends only on the metrics along the fibre and on M. For our convenience, we may consider an Ω which lies in a coordinate neighborhood, and then we may just use $\Omega, \partial\Omega$ instead of $\Omega \cap U$ and $\partial\Omega \cap U$.

The general first order differential operator in the Spencer sequence has the local form (1.1) with variable coefficients in Ω. Using (4.1) we find

$$(4.3) \qquad D^* = A_0^* + \sum_j \left(\sigma_j A_j^* - \frac{\partial A_j^*}{\partial x_j} - A_j^* \delta_j \right)$$

with boundary condition (1.4). From (4.3) and (1.5) a computation gives

$$
\begin{aligned}
{}^{\Omega}\|D^*u\|_\varphi^2 &= {}^{\Omega}\langle DD^*u, u\rangle_\varphi = -\sum_{i,j} {}^{\Omega}\langle A_i\delta_j\partial_i A_j^* u, u\rangle_\varphi \\
&\quad + \sum_{i,j} {}^{\Omega}\Big\langle \frac{\partial^2\varphi}{\partial x_i\partial x_j} A_i^* u, A_j^* u\Big\rangle_\varphi \\
&\quad + \sum_{i,j} {}^{\Omega}\langle A_i\partial_i\sigma_j A_j^* u, u\rangle_\varphi + {}^{\Omega}\langle Pu, u\rangle_\varphi ,
\end{aligned}
\tag{4.4}
$$

where

$$
P = -\sum_j A_0 A_j^*\delta_j + \sum_j A_0\sigma_j A_j^* + \sum_j A_j\partial_j A_0^* + A_0 A_0^* - \sum_j A_0\partial A_j^*/\partial x_j .
$$

We shall do the integration by parts for the first term on the right hand side of (4.4), and then re-express its boundary term:

$$
\begin{aligned}
&-\sum_{i,j} {}^{\Omega}\langle A_i\delta_j\partial_i A_j^* u, u\rangle_\varphi \\
&\quad = -\sum_{i,j} {}^{\Omega}\langle \delta_j A_i A_j^*\partial_i u, u\rangle_\varphi + {}^{\Omega}\langle P'u, u\rangle_\varphi \\
&\quad = \sum_{i,j} {}^{\Omega}\langle A_i A_j^*\partial_i u, \partial_j u\rangle_\varphi - \sum_{i,j} {}^{\Omega}\langle \sigma_j A_i A_j^*\partial_i u, u\rangle_\varphi + {}^{\Omega}\langle P'u, u\rangle_\varphi \\
&\qquad - \sum_{i,j} {}^{\partial\Omega}\langle \partial_i((\partial_j r)A_j^* u), A_i^* u\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\langle (\partial_i\partial_j r)A_j^* u, A_i^* u\rangle_\varphi \\
&\qquad + \sum_{i,j} {}^{\partial\Omega}\Big\langle (\partial_j r)\frac{\partial A_j^*}{\partial x_i} u, A_i^* u\Big\rangle_\varphi ,
\end{aligned}
\tag{4.5}
$$

where

$$
P' = \sum_{i,j} \frac{\partial A_i}{\partial x_j}\partial_i A_j^* - \sum_{i,j} \delta_j A_i \frac{\partial A_j^*}{\partial x_i} .
\tag{4.6}
$$

Now (1.4) implies that near the boundary $\sum_j (\partial_j r)A_j^* u = r(x)K(x)$; thus

$$
\begin{aligned}
-\sum_{i,j} {}^{\partial\Omega}\langle \partial_i((\partial_j r)A_j^* u), A_i^* u\rangle_\varphi &= -\sum_i {}^{\partial\Omega}\langle (\partial_i r)K, A_i^* u\rangle_\varphi \\
&\quad - \sum_i {}^{\partial\Omega}\langle r(\partial_i K), A_i^* u\rangle_\varphi = 0 .
\end{aligned}
\tag{4.7}
$$

On the other hand P and P' are first order differential operators containing δ_j which involves φ. We may get rid of these δ_j by replacing them by ∂_j through integration by parts. Altogether we have

$$
\begin{aligned}
{}^{\Omega}\|D^*u\|_\varphi^2 &= \sum_{i,j} {}^{\Omega}\langle A_i^*\partial_j u, A_j^*\partial_i u\rangle_\varphi + \sum_{i,j} {}^{\Omega}\Big\langle \frac{\partial^2\varphi}{\partial x_i\partial x_j} A_i^* u, A_j^* u\Big\rangle_\varphi \\
&\quad + \sum_{i,j} {}^{\partial\Omega}\Big\langle \frac{\partial^2 r}{\partial x_i\partial x_j} A_i^* u, A_j^* u\Big\rangle_\varphi + \sum_{i,j} {}^{\Omega}\Big\langle \frac{\partial\sigma_j}{\partial x_i} A_j^* u, A_i^* u\Big\rangle_\varphi \\
&\quad - {}^{\Omega}\langle P''u, u\rangle_\varphi ,
\end{aligned}
\tag{4.8}
$$

where

$$P'' = -2\Big(\sum_j A_j A_0^* \partial_j + \sum_{i,j} \frac{\partial A_j}{\partial x_i} A_i^* \partial_j\Big) - \Big(2 \sum_j A_j \frac{\partial A_0^*}{\partial x_j} + A_0 A_0^* + \sum_{i,j} \frac{\partial A_i}{\partial x_j} \frac{\partial A_j^*}{\partial x_i}\Big) . \tag{4.9}$$

Now (0.3) is true from our assumption that our operator satisfies the δ-estimate. If we write $|\partial\sigma/\partial x| = \sup_{i,j} |\partial\sigma_j/\partial x_i|$, and observe the obvious inequality

$$^{\Omega}\Big\|\sum_j B_j \partial_j u\Big\|_\varphi^2 \leq c[^{\Omega}\|Eu\|_\varphi^2 + {}^{\Omega}\|u\|_\varphi^2] , \tag{4.10}$$

then we have, by combining (4.8), (0.3) and (4.10),

$$\begin{aligned} &\sum_{i,j} {}^{\Omega}\Big\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^* u, A_j^* u\Big\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\Big\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u, A_j^* u\Big\rangle_\varphi \\ &\leq c_1\Big[^{\Omega}\|D^*u\|_\varphi^2 + {}^{\Omega}\|Eu\|_\varphi^2 + \Big|\frac{\partial\sigma}{\partial x}\Big| {}^{\Omega}\|u\|_\varphi^2\Big] \\ &\quad + [c_2\, {}^{\Omega}\|u\|_\varphi^2 + |{}^{\Omega}\langle P''u, u\rangle_\varphi|] , \end{aligned} \tag{4.11}$$

where c_1, c_2 are some constants independent of φ and u.

Note that if we replace D, E by D_ε and E_ε as defined by (2.1), we obtain similarly

$$\begin{aligned} &\sum_{i,j} {}^{\Omega}\Big\langle \frac{\partial^2\varphi}{\partial x_i \partial x_j} A_i^*(\varepsilon x) u, A_j^*(\varepsilon x) u\Big\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\Big\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^*(\varepsilon x) u, A_j^*(\varepsilon x) u\Big\rangle_\varphi \\ &\leq c_1\Big[^{\Omega}\|D_\varepsilon^*u\|_\varphi^2 + {}^{\Omega}\|E_\varepsilon u\|_\varphi^2 + \Big|\frac{\partial\sigma}{\partial x}\Big| {}^{\Omega}\|u\|_\varphi^2\Big] \\ &\quad + \varepsilon[c_2\, {}^{\Omega}\|u\|_\varphi^2 + |{}^{\Omega}\langle P_\varepsilon''u, u\rangle_\varphi|] , \end{aligned} \tag{4.11$_\varepsilon$}$$

which differs from (2.3) only by an extra term $|\partial\sigma/\partial x|\, {}^{\Omega}\|u\|_\varphi^2$, and P_ε'' is the same as expressed in formula (2.4).

The two lemmas in [6] used in § 2 in obtaining (2.5) are as follows. For each x_0 in Ω or in its boundary $\partial\Omega$, we can always find a neighborhood U of x_0 satisfiying the following lemmas.

Lemma 1. *There exists a constant c such that for all* $0 \leq \varepsilon \leq 1$,

$$|{}^{\Omega}\langle P_\varepsilon''u, u\rangle| \leq c[^{\Omega}\|D_\varepsilon^*u\|^2 + {}^{\Omega}\|E_\varepsilon u\|^2 + {}^{\Omega}\|u\|_{\frac{1}{2}}^2] ,$$

where

$$^{\Omega}\|u\|_{\frac{1}{2}}^2 = (2\pi)^{-n} \int_\Omega (1 + |\xi|^2)^{\frac{1}{2}} |\hat{u}(\xi)|\, d\xi$$

is the Sobolev norm.

Lemma 2. *There exists a constant c such that for all $\varepsilon > 0$ small enough we have*

$$^{\Omega}\|u\|_{\frac{1}{2}}^2 \leq c[^{\Omega}\|D_\varepsilon^* u\|^2 + {}^{\Omega}\|E_\varepsilon u\|^2 + {}^{\Omega}\|u\|^2 + {}^{\partial\Omega}\|u\|^2] .$$

Now P_ε'' can be written as $P_\varepsilon'' = \sum_j K_j\partial_j + \varepsilon K_0$; therefore, if we put $v = e^{-\frac{1}{2}\varphi}u$, we have

$$^{\Omega}\langle P_\varepsilon'' u, u\rangle_\varphi = \sum_j {}^{\Omega}\langle e^{-\frac{1}{2}\varphi}K_j\partial_j u, e^{-\frac{1}{2}\varphi}u\rangle + \varepsilon\, {}^{\Omega}\langle e^{-\frac{1}{2}\varphi}K_0 u, e^{-\frac{1}{2}\varphi}u\rangle$$
$$= {}^{\Omega}\langle P_\varepsilon'' v, v\rangle + \frac{1}{2}\sum_j {}^{\Omega}\left\langle \frac{\partial\varphi}{\partial x_j}K_j v, v\right\rangle .$$

Applying the above two lemmas and assuming (2.6) we obtain

$$|{}^{\Omega}\langle P_\varepsilon'' u, u\rangle_\varphi| \leq c[^{\Omega}\|D_\varepsilon^* v\|^2 + {}^{\Omega}\|E_\varepsilon v\|^2 + M\,{}^{\Omega}\|v\|^2 + {}^{\partial\Omega}\|v\|^2] . \tag{4.12}$$

However it can easily be verified that there exist some constants c_1, c_2 such that

$$\begin{aligned} {}^{\Omega}\|E_\varepsilon v\|^2 &\leq c_1[^{\Omega}\|E_\varepsilon u\|_\varphi^2 + M^2\,{}^{\Omega}\|u\|_\varphi^2] , \\ {}^{\Omega}\|D_\varepsilon^* v\|^2 &\leq c_2[^{\Omega}\|D_\varepsilon^* u\|_\varphi^2 + M^2\,{}^{\Omega}\|u\|_\varphi^2] . \end{aligned} \tag{4.13}$$

Combining (4.12), (4.13) we obtain immediately (2.5). The question now is how to shrink Ω in order to absorb the term $\varepsilon[c\,{}^{\Omega}\|u\|_\varphi^2 + |{}^{\Omega}\langle P_\varepsilon'' u, u\rangle_\varphi|]$.

We have already established the fact that in Fig. A

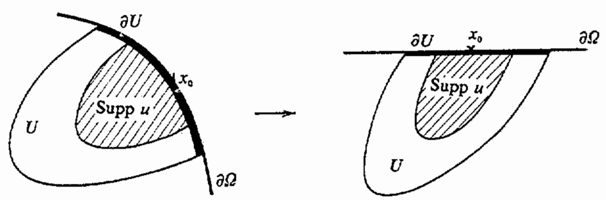

Fig. A Fig. B

there is a neighborhood U of x_0 in which (2.5) is true for u with support in U as indicated in the figure and satisfying the boundary condition. Now we shall flatten out the boundary by a suitable diffeomorphism and get Fig. B. Thus (2.5) (and of course (4.11)) is true in Fig. B. We may shrink the figure in Fig. B, that is, choose $\varepsilon > 0$ sufficiently small so that we can absorb the term $\varepsilon[c\,{}^{\Omega}\|u\|_\varphi^2 + |{}^{\Omega}\langle P_\varepsilon'' u, u\rangle_\varphi|]$ and get

$$\begin{aligned} &\sum_{i,j} {}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i\partial x_j}A_i^* u, A_j^* u\right\rangle_\varphi + \sum_{i,j} {}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i\partial x_j}A_i^* u, A_i^* u\right\rangle_\varphi \\ &\leq c[^{\Omega}\|D_\varepsilon^* u\|_\varphi^2 + {}^{\Omega}\|E_\varepsilon u\|_\varphi^2 + |\partial\sigma/\partial x|\,{}^{\Omega}\|u\|_\varphi^2] . \end{aligned} \tag{4.14}$$

If we make a coordinate transformation $y = \varepsilon x$, then D_ε^* is transformed into εD^*, E_ε into εE and $|\partial\sigma/\partial x|$ yields a factor ε^2 since σ_j, by (4.2), involves a first order differentiation. Thus in a sufficiently small ε-multiple of the domain in Fig. B we have the estimate

$$(4.15)\qquad \sum_{i,j}{}^{\Omega}\left\langle \frac{\partial^2\varphi}{\partial x_i\partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi + \sum_{i,j}{}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i\partial x_j} A_i^* u, A_j^* u\right\rangle_\varphi \leq c[{}^{\Omega}\|D^*u\|_\varphi^2 + {}^{\Omega}\|Eu\|_\varphi^2 + {}^{\Omega}\|u\|_\varphi^2] .$$

Thus by mapping diffeomorphically back to Fig. A we have proved

Theorem 6. *Let x_0 be a boundary point of Ω, and suppose that* (1.7) *and* (2.7) *are true. Then we can find a sufficiently small neighborhood U in which the estimate* (4.15) *holds for all u with support lying in U in the way described in Fig.* A, *where the constant c depends on φ and has the form* (2.8).

PART II

5. The D-Neumann problem

In [7] one can find the full description of the D-Neumann problem. We only point out here that if we can establish the Kohn-Nirenberg estimate (1.9), then the D-Neumann problem is solvable. Sweeney gave an example of a triangular operator for which the Kohn-Nirenberg estimate does not hold. However, if we assume that the operator satisfies the δ-estimate, then we shall prove that the Kohn-Nirenberg estimate is always true in a suitable convex domain. This we shall do in the next section.

6. The Kohn-Nirenberg estimate

Suppose that Ω with a smooth boundary $\partial\Omega$ is a subset in $\boldsymbol{R}^n$ or in a manifold. We shall consider the constant coefficient case first. In (4.8) and (4.9) we cancel all the terms involving the derivatives of the coefficients A_j, A_j^*, etc. We immediately get

$$(6.1)\qquad \sum_{i,j}{}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i\partial x_j} A_i^* u, A_j^* u\right\rangle \leq c[{}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2]$$

by setting $\varphi = 0$, where c is a constant independent of u but depending on the metrices. The left hand side is the Levi form and can be regarded as $H(\sigma(D)^*u)$ where H, the Hessian, is a bilinear form on $T^* \otimes C^{l-1}|_{\partial\Omega}$ defined as follows:

$$H\Big(\sum_i dx^i \otimes x_i,\ \sum_j dx^j \otimes y_j\Big) = \sum_{i,j}{}^{\partial\Omega}\langle(\partial_i\partial_j r)x_i, y_j\rangle .$$

For convex domains, the Hessian is positive definite on $T^*(\partial\Omega)\otimes C^{l-1}$ and the boundary condition $\sum_j (\partial_j r)A_j^* u = \sigma_{dr}(D)^* u \equiv 0$ on $\partial\Omega$ guarantees that $\sigma(D)^*u$ is in $T^*(\partial\Omega)\otimes C^{l-1}$, since the normal component vanishes. By also remembering that the symbol sequence of the Spencer sequence is surjective, we have

$$H(\sigma(D)^*u) \geq c\ {}^{\partial\Omega}\|\sigma(D)^*u\|^2 \geq c'\ {}^{\partial\Omega}\|u\|^2 .$$

Thus

$$\sum_{i,j} {}^{\partial\Omega}\left\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u, A_j^* u\right\rangle \geq c\ {}^{\partial\Omega}\|u\|^2$$

if Ω is Levi-convex, i.e., if the Levi form is positive definite. Consequently we have proved that if Ω is Levi convex, then the Kohn-Nirenberg estimate (1.9) is true.

For the Euclidean case, this is all which we want. But for the manifold case, we need the following partition of unity argument. Let us consider a compact manifold Ω, and assume that Ω is Levi-convex, that is, for every point of $\partial\Omega$ there is a Levi convex neighborhood. We have already shown that on this neighborhood the Kohn-Nirenberg estimate holds. Let us cover Ω with a finite number of the above mentioned Levi convex neighborhoods Ω_k for which the Kohn-Nirenberg estimate holds. We choose a partition of unity $\{\psi_k\}$ such that $\operatorname{Supp}\psi_k \subset \Omega_k$ and $\operatorname{Supp}\psi_k \cap \partial\Omega_k = \operatorname{Supp}\psi_k \cap \partial\Omega$. Then

$$\begin{aligned}
{}^{\partial\Omega}\|u\|^2 &\leq c\sum_k {}^{\partial\Omega}\|\psi_k u\|^2 \leq c\sum_k ({}^{\Omega}\|\psi_k u\|^2 + {}^{\Omega}\|D^*\psi_k u\|^2 + {}^{\Omega}\|E\psi_k u\|^2) \\
&\leq c\sum_k ({}^{\Omega}\|\psi_k u\|^2 + {}^{\Omega}\|\psi_k D^* u\|^2 + {}^{\Omega}\|\psi_k E u\|^2) \\
&\quad + c\sum_k ({}^{\Omega}\|[D^*, \psi_k]u\|^2 + {}^{\Omega}\|[E, \psi_k]u\|^2) \\
&\leq c({}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2) .
\end{aligned}$$

As for the variable coefficient case, we see from Theorem 6 that we are able to choose a finite covering $\{\Omega_k\}$ such that in each Ω_k Theorem 6 can be formulated. Take a similar partition of unity $\{\psi_k\}$, and assume that for all k

$$\sum_{i,j} {}^{\partial\Omega_k}\left\langle \frac{\partial^2 r}{\partial x_i \partial x_j} A_i^* u_k, A_j^* u_k\right\rangle \geq c\ {}^{\partial\Omega_k}\|u_k\| , \tag{6.2}$$

where $u_k = \psi_k u$ satisfies the boundary condition in each Ω_k. Thus the Kohn-Nirenberg estimate holds for each Ω_k, and we can apply the partition of unity argument as in the constant coefficient case to prove that under the assumption (6.2) the estimate is true in the whole region.

The above condition (6.2) is not very good, since it depends on the coordinate systems. Let us try to find an invariant condition to replace (6.2). From (4.9) we see that (4.8) contains a term $2\sum_{i,j}{}^{\Omega}\langle(\partial A_j/\partial x_i)A_i^*\partial_j u, u\rangle_\varphi$. Keep this term on the same side of the term $\sum_{i,j}{}^{\partial\Omega}\langle(\partial_i\partial_j r)A_i^* u, A_j^* u\rangle_\varphi$ in the derivation of formula (4.11), and then set $\varphi = 0$. The result is as follows:

$$\text{(6.3)}\qquad \sum_{i,j}{}^{\partial\Omega}\langle(\partial_i\partial_j r)A_i^* u, A_j^* u\rangle + 2\sum_{i,j}{}^{\Omega}\left\langle A_i^*\partial_j u, \frac{\partial A_j^*}{\partial x_i} u\right\rangle \le [{}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2] .$$

Now observe that by integration by parts

$$\sum_{i,j}{}^{\Omega}\left\langle A_i^*\partial_j u, \frac{\partial A_j}{\partial x_i} u\right\rangle = -\sum_{i,j}{}^{\Omega}\left\langle A_i^* u, \frac{\partial A_j^*}{\partial x_i}\partial_j u\right\rangle + \sum_{i,j}{}^{\partial\Omega}\left\langle A_i^* u, (\partial_j r)\frac{\partial A_j^*}{\partial x_i} u\right\rangle - \sum_{i,j}{}^{\Omega}\left\langle A_i^* u, \frac{\partial^2 A_j^*}{\partial x_i \partial x_j} u\right\rangle - \sum_{i,j}{}^{\Omega}\left\langle \frac{\partial A_i^*}{\partial x_j} u, \frac{\partial A_j^*}{\partial x_i} u\right\rangle .$$

Since the last two terms are of order $c\,{}^{\Omega}\|u\|^2$ and can be transported to the right hand side, the left hand side of (6.3) becomes

$$\sum_{i,j}{}^{\partial\Omega}\langle A_j(\partial_j\partial_i r)A_i^* u, u\rangle + \sum_{i,j}{}^{\partial\Omega}\left\langle A_j(\partial_i r)\frac{\partial A_i^*}{\partial x_j} u, u\right\rangle + \sum_{i,j}{}^{\Omega}\left\langle A_i^*\partial_j u, \frac{\partial A_j^*}{\partial x_i} u\right\rangle - \sum_{i,j}\left\langle A_i^* u, \frac{\partial A_j^*}{\partial x_i}\partial_j u\right\rangle$$
$$= \sum_{i,j}{}^{\partial\Omega}\langle A_j\partial_j((\partial_i r)A_i^*)u, u\rangle + \sum_{i,j}{}^{\Omega}\left\langle\left(\frac{\partial A_j}{\partial x_i}A_i^* - A_i\frac{\partial A_j^*}{\partial x_i}\right)\partial_j u, u\right\rangle ,$$

so that (6.3) becomes

$$\text{(6.4)}\qquad {}^{\partial\Omega}\langle(D\sigma_{dr}(D)^*)u, u\rangle + \sum_{i,j}{}^{\Omega}\left\langle\left(\frac{\partial A_j}{\partial x_i}A_i^* - A_i\frac{\partial A_j^*}{\partial x_i}\right)\partial_j u, u\right\rangle \le c[{}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2] .$$

We shall see more clearly that the Levi form, if it exists, must consist of the boundary integral ${}^{\partial\Omega}\langle D\sigma_{dr}(D)^*u, u\rangle$ and some boundary integrals, which we shall call the unknown part of the Levi form, contributed from the second term

$$\text{(6.5)}\qquad \sum_{i,j}{}^{\Omega}\left\langle\left(\frac{\partial A_j}{\partial x_i}A_i^* - A_i\frac{\partial A_j^*}{\partial x_i}\right)\partial_j u, u\right\rangle .$$

Fortunately by the combination of Lemma 1 ($\varepsilon = 1$) and a stronger version of Lemma 2 of § 4, which states that if we consider D, E instead of $D_\varepsilon, E_\varepsilon$, then Lemma 2 is still true [7], we see that (6.5) satisfies

$$\sum_{i,j}{}^{\Omega}\left\langle\left(\frac{\partial A_j}{\partial x_i}A_i^* - A_i\frac{\partial A_j^*}{\partial x_i}\right)\partial_j u, u\right\rangle \tag{6.6}$$
$$\leq c\,[{}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2 + {}^{\partial\Omega}\|u\|^2]\,,$$

so that the unknown part of the Levi form (if it exists) will be of the order $c\,{}^{\partial\Omega}\|u\|^2$. Thus, if we denote the Levi form by L, then

$${}^{\partial\Omega}\langle D\sigma_{d_r}(D)^*u, u\rangle - c\,{}^{\partial\Omega}\|u\|^2 \leq L \leq {}^{\partial\Omega}\langle D\sigma_{d_r}(D)^*u, u\rangle + c\,{}^{\partial\Omega}\|u\|^2\,.$$

The condition for L to be positive definite can be obtained by

$${}^{\partial\Omega}\langle D\sigma_{d_r}(D)^*u, u\rangle \geq K\,{}^{\partial\Omega}\|u\|^2 \tag{6.7}$$

for some sufficiently large positive constant K, and we obtain the Kohn-Nirenberg estimate from (6.7) in any case, even if L does not exist.

The left hand side of (6.7) is an invariant form. Therefore it can be interpreted globally, and in a local coordinate neighborhood we have the relation

$${}^{\partial\Omega}\langle D\sigma_{d_r}(D)^*u, u\rangle = \sum_{i,j}{}^{\partial\Omega}\left\langle\frac{\partial^2 r}{\partial x_i\partial x_j}A_i^*u, A_j^*u\right\rangle \tag{6.8}$$
$$+ \sum_{i,j}{}^{\partial\Omega}\left\langle A_j(\partial_i r)\frac{\partial A_i^*}{\partial x_j}u, u\right\rangle\,.$$

Thus the condition (6.7) for sufficiently large K is enough to guarantee the condition (6.2), and therefore we obtain

Theorem 7. *The Kohn-Nirenberg estimate holds on Ω if it satisfies* (6.7) *for u satisfying the boundary condition $\sigma_{d_r}(D)^*u \equiv 0$ on $\partial\Omega$.*

The first term in the expression (6.8) is called the dominant part, while the second term is called the secondary part. The reason is obvious when we consider D_ε, the dominant part remains unchanged, while the secondary part can be absorbed because of the appearance of a factor ε. In other word, if we are only interested in a small neighborhood Ω_k of x_0, then the secondary part can be neglected, and locally the condition (6.7) turns out to be our desired condition (6.2).

7. The Levi form and a remark concerning the $\bar{\partial}$-estimate

The guess work for the Levi form in the previous section is justified if we consider the special case where the operator is the $\bar{\partial}$-operator. In the computation for the $\bar{\partial}$-operator (we omit the details) the Levi form is obtained by omitting all terms, which can be absorbed into $\{{}^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Eu\|^2 + {}^{\Omega}\|u\|^2\}$, and then putting all the remaining boundary integrals together. We still do not know whether the Levi form of a general operator D in the Spencer sequence exists or not. In fact we shall give a simple example which raises some doubts about the existence of the Levi form.

Let us consider the following Dolbault complex on C^n

$$(7.1)\qquad 0 \longrightarrow \mathcal{O} \xrightarrow{\bar\partial} A^0 \xrightarrow{\bar\partial} A^1 \xrightarrow{\bar\partial} A^2 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} A^n \longrightarrow 0 .$$

By a bi-differentiable but nonanalytic transformation, (7.1) will be transformed into

$$(7.2)\qquad 0 \longrightarrow \tilde{\mathcal{O}} \xrightarrow{\tilde\partial} \tilde A^0 \xrightarrow{\tilde\partial} \tilde A^1 \xrightarrow{\tilde\partial} \tilde A^2 \xrightarrow{\tilde\partial} \cdots \xrightarrow{\tilde\partial} \tilde A^n \longrightarrow 0 .$$

Taking the direct sum of the two complexes we have

$$(7.3)\qquad 0 \to \mathcal{O} \oplus \tilde{\mathcal{O}} \xrightarrow{\bar\partial \oplus \tilde\partial} A^0 \oplus \tilde A^0 \xrightarrow{\bar\partial \oplus \tilde\partial} A^1 \oplus \tilde A^1 \xrightarrow{\bar\partial \oplus \tilde\partial} \cdots \xrightarrow{\bar\partial \oplus \tilde\partial} A^n \oplus \tilde A^n \to 0 .$$

The existence domains for solutions of (7.3) consist of domains in C^n whose convexity is defined by two Levi forms. For example, if we consider a unit ball B, we have the usual Levi form to express the convexity of B. However, we know that there exists a bi-differentiable mapping f such that $f(B) \subset C^n$ is no longer convex with respect to the original coodinate system. Thus to solve the Neumann problem we require two Levi forms, instead of a single Levi form, to yield the convexity condition on the bounded domain considered.

Finally, let us remark that in our theory we use the δ-estimate to control a certain term (0.3). We shall now give an example showing that we can do the same thing by a different kind of assumption. Let us consider the following complex

$$E^0 \xrightarrow{D} E^k \xrightarrow{D} E^{\frac{1}{2}k(k-1)}$$

where E^0 is a line bundle. Let the operator D be defined by

$$(7.4)\qquad Du = (p^1u, p^2u, \cdots, p^ku) .$$

Then the adjoint operator D^* is defined by

$$(7.5)\qquad D^*v = \sum_j p^{j*}v_j .$$

Therefore

$$\|D^*v\|^2 = \langle D^*v, D^*v\rangle = \langle DD^*v, v\rangle = \sum_{i,j} \langle p^i p^{j*} v_j, v_i\rangle$$
$$= \sum_{i,j} \langle p^{j*} p^i v_j, v_i\rangle + \sum_{i,j} \langle [p^i, p^{j*}] v_j, v_i\rangle .$$

Usually, the commutator $[p^i, p^{j*}]$ is a second order operator. However, if we make the assumption that the principal symbols commute with the symbols of the adjoint operators, then the commutator $[p^i, p^{j*}]$ becomes a first order operator, and we have

$$(7.6) \quad \|D^*v\|^2 = \sum_{i,j} \langle p^i v_j, p^j v_i \rangle + \text{commutator term} + \text{boundary term}.$$

But we have also

$$(7.7) \qquad \|Dv\|^2 = -\sum_{i,j} \langle p^i v_j, p^j v_i \rangle + \sum_{i,j} \|p^i v_j\|^2 .$$

Combining (7.6) and (7.7) we obtain

$$\|Dv\|^2 + \|D^*v\|^2 = \text{commutator term} + \sum_{i,j} \|p^i v_j\|^2 + \text{boundary term} ,$$

and then we can follow our standard technique to derive our estimates. Note that, by a theorem of Mackichan, the δ-estimate implies that the principal symbols commute with the symbols of the adjoint operator. However, it is not known whether the converse is true or not.

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